Anyonic behavior of quantum group gases

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We first introduce and discuss the formalism of $SU_q(N)$ bosons and fermions and consider the simplest Hamiltonian involving these operators. We then calculate the grand partition function for these models and study the high temperature (low density) case of the corresponding gases for $N=2$. We show that quantum group gases exhibit anyonic behavior in $D=2$ and 3 spatial dimensions. In particular, for a $SU_q(2)$ boson gas at $D=2$ the parameter q interpolates within a wider range of attractive and repulsive systems than the anyon statistical parameter. [S1063-651X(97)11301-0]

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I. INTRODUCTION

In the last few years, the search for new applications of quantum groups and quantum algebras [1,2], other than the theory of integrable models and the quantum inverse scattering method, has attracted the attention of mathematicians and physicists alike. The published literature on formulations based on quantum group theory includes studies in noncommutative geometry [3,4], quantum mechanics [5], field theory [6], and molecular and nuclear physics [7]. Many of these approaches are attempts to develop more general formulations of quantum mechanics and field theory, and to look for small deviations from the standard value $q=1$ in nuclear and molecular physics. In this paper we study the high temperature (low density) behavior of two quantum group gases. In Sec. II we discuss the covariant $su_q(N)$ fermion and boson algebras, and specialize to the case $N=2$. In Secs. II A and II B, we introduce the $SU_q(2)$ fermion and boson models respectively, and in each case we give a representation of these operators in terms of the corresponding standard fermion or boson oscillators. Section III contains the main results of this work. We obtain the equation of state as a virial expansion, and discuss their anyonic behavior for both gases at $D=2$ and 3. In $D=2$ we compare the parameter q with the anyon statistical parameter α.

II. QUANTUM GROUP BOSONS AND FERMIONS

In this section we briefly discuss the quantum group field algebras introduced in Ref. [8]. These algebras can be seen as generalizations of the standard bosonic and fermionic algebras. As it is well known, bosonic and fermionic operators satisfy the algebraic relations

$$\phi_i\phi_j^\dagger-\phi_j^\dagger\phi_i=\delta_{ij},$$
$$\psi_i\psi_j^\dagger+\psi_j^\dagger\psi_i=\delta_{ij}, \tag{1}$$

which, for $i,j=1,\ldots N$, are covariant under $SU(N)$ transformations. The quantum group analogs of these equations are given by the relations

$$\Omega_j\overline{\Omega_i}=\delta_{ij}\pm q^{\pm1}R_{kijl}\overline{\Omega_l}\Omega_k, \tag{2}$$

$$\Omega_l\Omega_k=\pm q^{\mp1}R_{jikl}\Omega_j\Omega_i, \tag{3}$$

where $\Omega=\Phi$ and Ψ, and the upper (lower) sign applies to quantum group bosons Φ_i (quantum group fermions Ψ_i) operators. The $N^2\times N^2$ matrix R_{jikl} is explicitly written [4]

$$R_{jikl}=\delta_{jk}\delta_{il}(1+(q-1)\delta_{ij})+(q-q^{-1})\delta_{ik}\delta_{jl}\theta(j-i), \tag{4}$$

where $\theta(j-i)=1$ for $j>i$ and zero otherwise. Denoting the new fields as $\Omega_i'=\Sigma_{i=1}^N T_{ij}\Omega_j$, the $SU_q(N)$ transformation matrix T and the R matrix satisfy the well known algebraic relations [9]

$$RT_1T_2=T_2T_1R, \tag{5}$$

and

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}, \tag{6}$$

with the standard embedding $T_1=T\otimes 1$, $T_2=1\otimes T\in V\otimes V$ and $(R_{23})_{ijk,i'j'k'}=\delta_{ii'}R_{jk,j'k'}\in V\otimes V\otimes V$.

In particular, for $N=2$, Eqs. (2) and (3) are simply written as follows.

(a) $SU_q(2)$—fermions:

$$\{\Psi_2,\overline{\Psi_2}\}=1, \tag{7}$$

$$\{\Psi_1,\overline{\Psi_1}\}=1-(1-q^{-2})\overline{\Psi_2}\Psi_2, \tag{8}$$

$$\Psi_1\Psi_2=-q\Psi_2\Psi_1, \tag{9}$$

$$\overline{\Psi}_1\Psi_2=-q\Psi_2\overline{\Psi}_1, \tag{10}$$

$$\{\Psi_1,\Psi_1\}=0=\{\Psi_2,\Psi_2\}. \tag{11}$$

(b) $SU_q(2)$—bosons:

$$\Phi_2\overline{\Phi_2}-q^2\overline{\Phi_2}\Phi_2=1, \tag{12}$$

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$$\Phi_1\overline{\Phi}_1-q^2\overline{\Phi}_1\Phi_1=1+(q^2-1)\overline{\Phi}_2\Phi_2, \tag{13}$$

$$\Phi_2\Phi_1=q\Phi_1\Phi_2, \tag{14}$$

$$\Phi_2\overline{\Phi}_1=q\overline{\Phi}_1\Phi_2, \tag{15}$$

which for $q=1$ become the fermion and boson algebras, respectively. According to Eq. (5) the matrix $T=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ elements generate the algebra

$$\begin{gathered} ab=q^{-1}ba, \quad ac=q^{-1}ca, \\ bc=cd, \quad dc=qcd, \\ db=qbd, \quad da-ad=(q-q^{-1})bc, \\ \det_q T\equiv ad-q^{-1}bc=1, \end{gathered} \tag{16}$$

with the unitary conditions [10] $\overline{a}=d$, $\overline{b}=q^{-1}c$, and $q\in\mathbb{R}$. Hereafter, we take $0\leqslant q<\infty$.

A. SU$_q$(2) fermion model

The simplest Hamiltonian one can write in terms of the operators Ψ_i is simply

$$\mathcal{H}_F=\sum_\kappa \varepsilon_\kappa(\mathcal{M}_{1,\kappa}+\mathcal{M}_{2,\kappa}), \tag{17}$$

where $\mathcal{M}_{i\kappa}=\overline{\Psi}_{i,\kappa}\Psi_{i,\kappa}$ and $\{\overline{\Psi}_{\kappa,i},\Psi_{\kappa',j}\}=0$ for $\kappa\neq\kappa'$. From Eq. (11) we see that the occupation numbers are restricted to $m=0$ and 1, and therefore SU$_q(N)$ fermions satisfy the Pauli exclusion principle. For a given κ, a normalized state is simply written

$$\overline{\Psi}_2^n\overline{\Psi}_1^m|0\rangle, \quad n,m=0,1, \tag{18}$$

and the $\mathcal{M}_i$ operator satisfies

$$[\mathcal{M}_2,\Psi_1]=0 \tag{19}$$

and

$$\mathcal{M}_1\Psi_2-q^2\Psi_2\mathcal{M}_1=0. \tag{20}$$

The grand partition function is given by

$$\mathcal{Z}_F=\mathrm{Tr}e^{-\Sigma_\kappa\varepsilon_\kappa(\mathcal{M}_{1,\kappa}+\mathcal{M}_{2,\kappa})}e^{\beta\mu(M_{1,\kappa}+M_{2,\kappa})}, \tag{21}$$

where $M_{i,\kappa}=\psi^\dagger_{i,\kappa}\psi_{i,\kappa}$ are the standard fermion number operators, and the trace is taken with respect to the states in Eq. (18). Since the pair $\Psi_2,\overline{\Psi}_2$ satisfies standard anticommutation relations, we can identify it without any loss of generality with a fermion pair $\psi_2,\psi_2^\dagger$. In addition, from Eqs. (8) and (11) we see that the operator $\Psi_1(\overline{\Psi}_1)$ is a function of the operator $\psi_1(\psi_1^\dagger)$ times a function of $\mathcal{M}_2$. Therefore the grand partition function $\mathcal{Z}_F$ becomes

$$\mathcal{Z}_F=\prod_\kappa\sum_{n=0}^{1}\sum_{m=0}^{1} e^{-\beta\varepsilon_\kappa(n+m-(1-q^{-2})mn}e^{\beta\mu(n+m)} \tag{22}$$

$$=\prod_\kappa (1+2e^{-\beta(\varepsilon_\kappa-\mu)}+e^{-\beta(\varepsilon_\kappa(q^{-2}+1)-2\mu)}), \tag{23}$$

which for $q=1$ becomes the square of a single-fermion-type grand partition function. From Eq. (23) we see that the original Hamiltonian becomes the interacting Hamiltonian

$$H_F=\sum_\kappa \varepsilon_\kappa(M_{1,\kappa}+M_{2,\kappa}+(q^{-2}-1)M_{1,\kappa}M_{2,\kappa}). \tag{24}$$

Therefore the parameter $q\neq1$ mixes the two degrees of freedom in a nontrivial way through a quartic term in the Hamiltonian. The thermodynamics of this system will be discussed in Sec. III A.

A simple check shows that Eqs. (8)–(11) and (24) are consistent with the following representation of Ψ_i operators in terms of fermion operators ψ_j:

$$\Psi_2=\psi_2, \tag{25}$$

$$\overline{\Psi}_2=\psi_2^\dagger, \tag{26}$$

$$\Psi_1=\psi_1(1+(q^{-1}-1)M_2), \tag{27}$$

$$\overline{\Psi}_1=\psi_1^\dagger(1+(q^{-1}-1)M_2), \tag{28}$$

and, according to Eqs. (2) and (3), this result easily generalizes for arbitrary N to

$$\Psi_m=\psi_m\prod_{l=m+1}^{N}(1+(q^{-1}-1)M_l), \tag{29}$$

and similarly for the adjoint equation.

It is interesting to remark the distinction between SU$_q$(2) fermions with the so called q-fermions b_i and $b_i^\dagger$. The q-fermionic algebra was introduced in [11]

$$bb^\dagger+qb^\dagger b=q^{N_q}, \tag{30}$$

$$b^\dagger b=[N_q], \tag{31}$$

$$bb^\dagger=[1-N_q], \tag{32}$$

$$b^2=0=b^{\dagger 2}, \tag{33}$$

where the bracket $[x]=(q^x-q^{-x})/(q-q^{-1})$ and the number operator $N_q|n\rangle=n|n\rangle$ with $n=0$ and 1. Since the q-number $[x]=x$ for $x=0$ and 1, it is obvious that the grand partition function for q-fermions is no different than the Fermi grand partition function, and therefore the q-fermions do not lead to new results as far as thermodynamics is concerned.

B. SU$_q$(2) boson model

In terms of SU$_q$(2) bosons, we introduce the following Hamiltonian:

$$\mathcal{H}_B=\sum_\kappa \varepsilon_\kappa(\mathcal{N}_{1,\kappa}+\mathcal{N}_{2,\kappa}), \tag{34}$$

where $[\overline{\Phi}_{i,k},\Phi_{\kappa',j}]=0$ for $\kappa\neq\kappa'$. The operator $\mathcal{N}_{i,\kappa}=\overline{\Phi}_{i,\kappa}\Phi_{i,\kappa}$ satisfies the relations

$$[\mathcal{N}_{2,\kappa},\Phi_1]=0 \tag{35}$$

and

$$\mathcal{N}_{1,\kappa}\Phi_2-q^{-2}\Phi_2\mathcal{N}_{1,\kappa}=0. \tag{36}$$

The states are built by the action of the Φ operators on the vacuum state. For example, for a given κ a normalized state with n_1 particles of species 1 and n_2 particles of species 2 is defined by

$$\frac{1}{\sqrt{\{n_1\}!\{n_2\}!}}\overline{\Phi}_2^{n_2}\overline{\Phi}_1^{n_1}|0\rangle, \tag{37}$$

where the q-numbers $\{n\}=(1-q^{2n})/(1-q^2)$ and the q-factorials $\{n\}!$ are defined $\{n\}!=\{n\}\{n-1\}\{n-2\}\cdots 1$. The grand partition function $\mathcal{Z}_B$ is written

$$\mathcal{Z}_B=\mathrm{Tr}e^{-\beta\varepsilon_\kappa(\overline{\Phi}_{1,\kappa}\Phi_{1,\kappa}+\overline{\Phi}_{2,\kappa}\Phi_{2,\kappa})}e^{-\beta\mu(N_{1,\kappa}+N_{2,\kappa})}, \tag{38}$$

where $N_{i,\kappa}$ are the ordinary boson number operators $N_{i,\kappa}=\phi^\dagger_{i,\kappa}\phi_{i,\kappa}$ and the trace is taken with respect to the states in Eq. (37). For a given κ the $\mathrm{SU}_q(2)$ bosons are written in terms of boson operators $\phi_{i,\kappa}$ and $\phi^\dagger_{i,\kappa}$ with usual commutations relations $[\phi_i,\phi_j^\dagger]=\delta_{ij}$ as follows:

$$\Phi_2=(\phi_2^\dagger)^{-1}\{N_2\}, \tag{39}$$

$$\overline{\Phi}_2=\phi_2^\dagger, \tag{40}$$

$$\Phi_1=(\phi_1^\dagger)^{-1}\{N_1\}q^{N_2}, \tag{41}$$

$$\overline{\Phi}_1=\phi_1^\dagger q^{N_2}. \tag{42}$$

The grand partition function $\mathcal{Z}_B$ then becomes

$$\mathcal{Z}_B=\prod_\kappa\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}e^{-\beta\varepsilon_\kappa\{n+m\}}e^{\beta\mu(n+m)}, \tag{43}$$

with the corresponding interacting Hamiltonian

$$H_B=\sum_\kappa \varepsilon_\kappa\{\phi^\dagger_{1,\kappa}\phi_{1,\kappa}+\phi^\dagger_{2,\kappa}\phi_{2,\kappa}\}, \tag{44}$$

with the bracket $\{x\}$ as defined below Eq. (37). Therefore, the original Hamiltonian becomes a Hamiltonian in terms of ordinary boson interactions involving powers of the number operators $N_{i,\kappa}$ and $\ln q$. Equations (39)–(42) are easily generalized for $N>2$ to the set of equations

$$\overline{\Phi}_m=\phi_m^\dagger\prod_{l=m+1}^{N}q^{N_l} \tag{45}$$

and

$$\Phi_m=(\phi_m^\dagger)^{-1}\{N_m\}\prod_{l=m+1}^{N}q^{N_l}, \tag{46}$$

and a $\mathrm{SU}_q(N)$ boson state in terms of boson operators reads

$$\frac{1}{\sqrt{\{n_1\}!\{n_2\}!\cdots\{n_M\}!}}\phi^{\dagger n_M}_{M,\kappa_M}\phi^{\dagger n_{M-1}}_{M-1,\kappa_{M-1}}\cdots\phi^{\dagger n_1}_{1,\kappa_1}|0\rangle. \tag{47}$$

The normalization is consistent with the fact that the dual of the state in Eq. (47) is obtained by applying the adjoint operation defined on Φ. The number operator $N_l=\phi_l^\dagger\phi_l$ satisfies standard commutation relations with the operators Φ_m,

$$[N_{l,\kappa},\overline{\Phi}_{m,\kappa'}]=\overline{\Phi}_{m,\kappa'}\delta_{\kappa,\kappa'}\delta_{l,m} \tag{48}$$

and

$$[N_{l,\kappa},\Phi_{m,\kappa'}]=-\Phi_{m,\kappa}\delta_{\kappa,\kappa'}\delta_{l,m}, \tag{49}$$

such that

$$N_l\overline{\Phi}_l^m|0\rangle=m\overline{\Phi}_l^m|0\rangle. \tag{50}$$

The difference between the operators Φ and the so called q-bosons is obvious. A set $(a_i,a_i^\dagger)$ of q-bosons satisfies the relations [13,14]

$$a_ia_i^\dagger-q^{-1}a_i^\dagger a_i=q^N,\quad [a_i,a_j^\dagger]=0=[a_i,a_j], \tag{51}$$

where $N|n\rangle=n|n\rangle$. By taking two commuting sets of q bosons, it has been shown [11] that the operators

$$J_+=a_2^\dagger a_1,\quad J_-=a_1^\dagger a_2,\quad 2J_3=N_2-N_1 \tag{52}$$

generate the quantum Lie algebra $\mathrm{su}_q(2)$

$$[J_3,J_\pm]=\pm J_\pm,\quad [J_-,J_-]=[2J_3]. \tag{53}$$

In contrast to the algebraic relations involving the operators Φ_i and Φ_j, Eq. (51) with $i,j=1,2$ is not covariant under the action of the $\mathrm{SU}_q(2)$ quantum group matrices. The thermodynamics of q-bosons and similar operators called quons [12] has been studied by several authors [15,16]. In Sec. III we study the thermodynamics of the two $\mathrm{SU}_q(2)$ models described in this section.

III. QUANTUM GROUP GASES

The high and low temperature behaviors of the $\mathrm{SU}_q(2)$ fermion model have been studied in Refs. [17, 18], and here we recall some results that will be compared with the $\mathrm{SU}_q(2)$ boson case.

A. Quantum group fermion gas

The internal energy U for this model is calculated from the grand potential $\Omega=(-1/\beta)\ln\mathcal{Z}_F$ according to the equation

$$U=\left(\frac{\partial\beta\Omega}{\partial\beta}+\mu M\right)=V\int\frac{p^2}{2m}\frac{(2+(q^{-2}+1)e^{\beta(\mu-[(q^{-2}p^2)/2m])}d^3p}{(2\pi\hbar)^3f(\varepsilon,\mu,q)}, \tag{54}$$

where the function $f(\varepsilon,\mu,q)=e^{\beta(\varepsilon-\mu)}+2+e^{-\beta(q^{-2}\varepsilon-\mu)}$. The low temperature regime of a $SU_q(2)$ fermion gas exhibits the interesting feature that for every value of $q\neq 1$ the entropy lies below the Fermi entropy. For $q>1$ and $q<1$ the entropy functions are given, respectively, by the equations [18]

$$S(q>1)\approx\lambda\,\frac{1.28\sqrt{2\mu_0}k^2T}{(q^{-2}+1)^{3/2}} \tag{55}$$

and

$$S(q<1)\approx\lambda k^2\sqrt{\mu_0}T\left[1.08(q^3+1)-\frac{(1-q^3)^2}{2(1+q^3)}\ln^2 3\right], \tag{56}$$

where $\lambda=[4\pi V(2m)^{3/2}/(2\pi\hbar)^3]$. The lower bound to the entropy values corresponds to the limit $q\to 0$. Furthermore, systems described by a Hamiltonian with $q>1$ share the same entropy function with systems with $q<1$. Comparing Eq. (55) with Eq. (56), we obtain that two gases share the entropy function if the following relation is satisfied:

$$(1+q'^{-2})^{3/2}=\frac{3.62(1+q^3)}{2.16(1+q^3)^2-(1-q^3)^2\ln^2 3}, \tag{57}$$

where $q'>1$ and $q<1$. Specifically, the equality is satisfied in the interval $0.33\leqslant q<0.91$.

The high temperature behavior of this model is also interesting. Starting with the grand partition function $\mathcal{Z}_F$,

$$\ln\mathcal{Z}_F=\frac{4\pi V}{h^3}\int_0^\infty p^2\ln(1+2e^{-\beta(\varepsilon-\mu)}+e^{-\beta(\varepsilon(q^{-2}+1)-2\mu)})dp, \tag{58}$$

it was shown in Ref. [18] that in $D=3$ spatial dimensions the virial expansion leads to the equation of state

$$pV=kT\langle M\rangle\left(1+\frac{\alpha(q)}{2}\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\frac{\langle M\rangle}{V}+\cdots\right), \tag{59}$$

where the coefficient $\alpha(q)=(1/2^{3/2})-[1/2(q^{-2}+1)^{3/2}]$. From Eq. (59) we see that the sign of the second virial coefficient depends on the value of q, showing that the parameter q interpolates between attractive and repulsive behavior. The function $\alpha/2$ takes values in the interval $2^{-5/2}\geqslant\alpha/2\geqslant-2^{-5/2}(\sqrt{2}-1)$ as q varies from zero to ∞, and vanishes at $q=1.96$. Figure 1 shows a graph of the function $B(q,T)=[\alpha(q)/2]\beta^{3/2}$ for large values of the temperature, and $q=10$, 1.96, 1, and 0.3.

It is important to remark that the free boson limit $B_b(T)=-2^{-7/2}\beta^{3/2}<B(\infty,T)=-2^{-5/2}(\sqrt{2}-1)\beta^{3/2}$, and therefore free bosons are not described in this model. A natural question to address is whether a similar interpolation occurs at $D=2$. The same procedure leads to the equation of state

$$pA=kT\langle M\rangle\left(1+\frac{1}{1+q^2}\,\frac{h^2}{8\pi mkT}\,\frac{\langle M\rangle}{A}+\cdots\right), \tag{60}$$

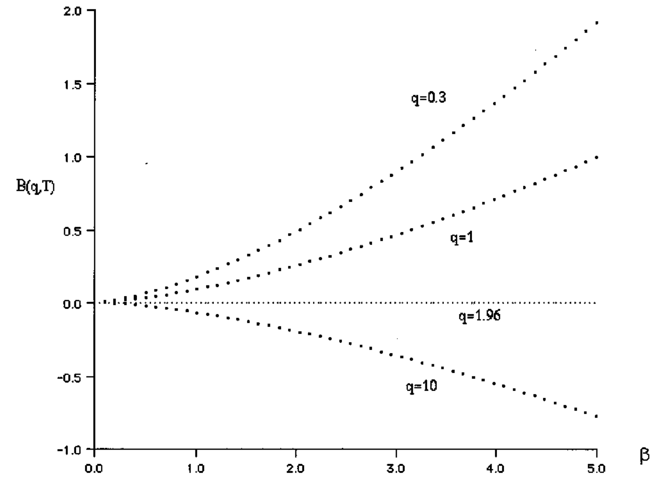

FIG. 1. The function $B(q,T)$ as defined in the text in the interval $0\leqslant\beta\leqslant 5$/eV and four values of q.

wherein the second virial coefficient is positive for all values of q, showing that this model, at $D=2$, describes only repulsive systems.

B. Quantum group boson gas

The grand partition function $\mathcal{Z}_B$ in Eq. (43) can be simply rewritten as

$$\mathcal{Z}_B=\prod_\kappa\sum_{m=0}^\infty(m+1)e^{-\beta\varepsilon_\kappa\{m\}}z^m, \tag{61}$$

where $z=e^{\beta\mu}$ is the fugacity. In $D=3$ the first few terms in powers of z read

$$\ln\mathcal{Z}_B=\frac{4\pi V}{h^3}\int_0^\infty dp\ p^2\Big(2e^{-\beta\varepsilon_\kappa}z+(6e^{-\beta\varepsilon_\kappa\{2\}}-4e^{-\beta\varepsilon_\kappa 2})\times\frac{z^2}{2}+(24e^{-\beta\varepsilon_\kappa\{3\}}-36e^{-\beta\varepsilon_\kappa\{2\}}e^{-\beta\varepsilon_\kappa}+16e^{-\beta\varepsilon_\kappa 3})\times\frac{z^3}{3!}+\cdots\Big), \tag{62}$$

such that performing the elementary integrations gives

$$\ln\mathcal{Z}_B=\frac{4\pi V}{h^3}\left(\frac{\sqrt{\pi}}{2}\left(\frac{2m}{\beta}\right)^{3/2}z+\sqrt{\pi}\left(\frac{2m}{\beta}\right)^{3/2}\delta(q)z^2+\cdots\right), \tag{63}$$

where $\delta(q)=\frac{1}{4}([3/(1+q^2)^{3/2}]-(1/\sqrt{2}))$.

Calculating the average number of particles $\langle N\rangle=(1/\beta)[(\partial\ln\mathcal{Z}_B/\partial\mu)]_{T,V}$, and reverting the equation, we find, for the fugacity,

$$z\approx\frac{1}{2}\left(\frac{h^2}{2m\pi kT}\right)^{3/2}\frac{\langle N\rangle}{V}-\delta(q)\left(\frac{h^2}{2m\pi kT}\right)^3\left(\frac{\langle N\rangle}{V}\right)^2, \tag{64}$$

leading to the following equation of state:

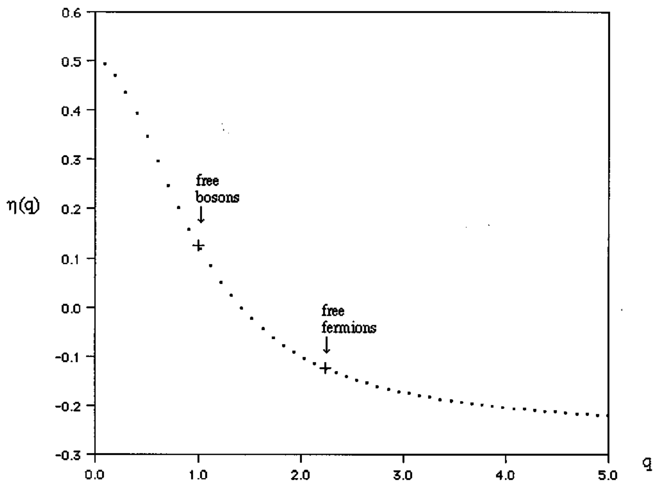

FIG. 2. The coefficient $\eta(q)$ for the interval $0\leqslant q\leqslant 5$. At the values $q=1$ and $5^{1/2}$ the system behaves as a free boson and fermion gas, respectively. The second virial coefficient vanishes at $q=2^{1/2}$.

$$pV=kT\langle N\rangle\left(1-\delta(q)\left(\frac{h^2}{2m\pi kT}\right)^{3/2}\frac{\langle N\rangle}{V}+\cdots\right). \quad (65)$$

As expected, at $q=1$ the coefficient $\delta(1)=2^{-7/2}$, which is the numerical factor in the second virial coefficient for a free boson gas with two species. The free fermion $\delta(q)=-2^{7/2}$ and ideal gas $\delta(q)=0$ cases are reached at $q\approx 1.78$ and $q\approx 1.27$, respectively.

A very similar calculation for $D=2$ gives the equation of state

$$pA=kT\langle N\rangle\left(1-\eta(q)\,\frac{h^2}{2\pi mkT}\,\frac{\langle N\rangle}{A}+\cdots\right), \quad (66)$$

with $\eta(q)=[(2-q^2)/4(1+q^2)]$. At $D=2$ this model behaves as a fermion gas for $q=\sqrt{5}$. Figure 2 shows a graph of the coefficient $\eta(q)$ as a function of the parameter q for $D=2$.

Since the $SU_q(2)$ boson gas at $D=2$ also interpolates completely between bosons and fermions, we can find a relation between the parameter q and the statistical parameter α for an anyon gas [19,20] of two species. This relation is given by

$$\alpha=1-\left(\frac{5-q^2}{2(1+q^2)}\right)^{1/2}, \quad (67)$$

where $0\leqslant\alpha\leqslant 1$, with the boson and fermion limits $\alpha=0$ ($q=1$) and $\alpha=1$ ($q=\sqrt{5}$), respectively. The second virial coefficient in Eq. (66) takes values in the interval $[-(\lambda_T^2/2),(\lambda_T^2/4)]$, with $\lambda_T=\sqrt{h^2/2\pi mkT}$, and therefore the parameter q interpolates within a larger range of systems than the α parameter does.

IV. CONCLUSIONS

In this paper we studied the high temperature behavior of quantum group gases. Our approach is mainly based on promoting the su(N) covariant fermion and boson algebras to the corresponding algebraic relations covariant under $SU_q(N)$ transformations. For purposes of simplicity we considered the $N=2$ case. Starting with the simplest Hamiltonian we calculated the partition function and obtained the equation of state for the two $SU_q(2)$ gases. Certainly, for $q=1$ our results become those for two species of free fermion or boson gases. For $q\neq 1$ this degeneracy is broken, and the corresponding Hamiltonian written in terms of standard operators acquires an interaction term. Our results indicate that the q parameter interpolates between repulsive and attractive behaviors. In particular, for a $SU_q(2)$ fermion gas and $D=3$, the sign of the second virial coefficient depends on the value of q. The ideal gas case corresponds to $q=1.96$ and the system becomes repulsive for $q<1.96$. For $q>1.96$ the system becomes attractive, but as $q\to\infty$ the free boson limit is not reached, and therefore this model does not interpolate completely between the free fermion and free boson cases. For $D=2$ the second virial coefficient of this gas is positive for every value of q and vanishes in the $q\to\infty$ limit. For $SU_q(2)$ bosons the results are more interesting. For $D=2$ and 3 the parameter q interpolates completely between a wide range of attractive and repulsive systems, including the free fermion and boson cases. For $D=2$ we found a relation between q and the statistical parameter α for an anyon gas. Therefore, the simple models studied here, and in particular the $SU_q(2)$ boson model, offer an alternative approach in describing systems obeying fractional statistics in two and three spatial dimensions.

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